

A FREE BOUNDARY PROBLEM ON THREE-DIMENSIONAL CONES

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ABSTRACT. We consider a free boundary problem on cones depending on a parameter c and study when the free boundary is allowed to pass through the vertex of the cone. We show that when the cone is three-dimensional and c is large enough, the free boundary avoids the vertex. We also show that when c is small enough but still positive, the free boundary is allowed to pass through the vertex. This establishes 3 as the critical dimension for which the free boundary may pass through the vertex of a right circular cone. In view of the well-known connection between area-minimizing surfaces and the free boundary problem under consideration, our result is analogous to a result of Morgan that classifies when an area-minimizing surface on a cone passes through the vertex.

1. INTRODUCTION

We study solutions to the problem

$$(1.1) \quad \begin{aligned} \Delta u &= 0 && \text{in } \{u > 0\} \\ |\nabla u| &= 1 && \text{on } \partial\{u > 0\}. \end{aligned}$$

on right circular cones in \mathbb{R}^n . We are interested in determining when the free boundary $\partial\{u > 0\}$ is allowed to pass through the vertex of the cone.

The above problem has applications to two dimensional flow problems as well as heat flow problems (see [4] where (1.1) was first studied). When considering the applications on a manifold, one studies a variable coefficient problem in divergence form:

$$(1.2) \quad \begin{aligned} \partial_j(a^{ij}(x)u_i) &= 0 && \text{in } \{u > 0\} \\ a^{ij}(x)u_i u_j &= 1 && \text{on } \partial\{u > 0\}. \end{aligned}$$

Solutions of (1.2) may be found inside a bounded domain Ω by minimizing the functional:

$$(1.3) \quad \int_{\Omega} a^{ij}(x)v_i v_j + Q(x)\chi_{\{v>0\}}.$$

However, since the functional is not convex, minimizers of (1.3) may not be unique and there exist solutions to (1.2) which are not minimizers of (1.3). When the coefficients $a^{ij}(x)$ are Lipschitz continuous and satisfy an ellipticity condition, regularity of the free boundary was studied in [15]. The authors in [15] adapted the sup-convolution approach of Caffarelli in [7–9] for viscosity solutions. This approach relies on a nondivergence structure and therefore requires Lipschitz continuity of the coefficients $a^{ij}(x)$ so that (1.3) can be transformed into a nondivergence operator.

More recently, the regularity of the free boundary for Hölder continuous coefficients $a^{ij}(x)$ was accomplished in [11] using different techniques. For coefficients $a^{ij}(x)$ assumed merely to be bounded, measurable, and satisfying the usual ellipticity conditions, regularity of the solution and its growth away from the free boundary was studied in [14]. However, to date nothing is known regarding the regularity of the free boundary when the coefficients $a^{ij}(x)$ are allowed to be discontinuous. In this paper we are interested in how the free boundary interacts with isolated discontinuous points of the coefficients $a^{ij}(x)$. In the context of a hypersurface, these points are considered to be a topological singularity. The simplest such case is the vertex of a cone. The aim of this paper is to study when the free boundary of a solution that arises as a minimizer is allowed to pass through a topological singularity. Before stating the main results of this paper we first recall a connection between solutions to (1.1) and minimal surfaces in order to understand what results one might expect for the free boundary problem on a cone.

1.1. Connection to minimal surfaces. Results for the singular set of free boundary points are analogous to results for the singular set of minimal surfaces. In the case of area-minimizing surfaces, the study of the singular set is reduced to considering area-minimizing cones. Simons [20] showed that any area-minimizing cone in \mathbb{R}^n for $n \leq 7$ is necessarily planar. Simons actually proved a stronger result in [20] by showing that any minimal stable cone is planar. He also provided an example of a cone in \mathbb{R}^8 that is stable and therefore a possible candidate for being an area-minimizing cone. One year later, it was shown that the Simons cone is indeed area-minimizing, see [6]. As a consequence, $n = 8$ is the first dimension for which a singularity of an area-minimizing hypersurface may occur.

Regarding the singular set of the free boundary for minimizers, the authors in [4] showed there are no singular points in dimension $n = 2$. In [22] a monotonicity formula is utilized to show that blow-up solutions are homogeneous, and therefore the free boundary of blow-up solutions is a cone. As a further consequence there exists a minimal dimension k^* such that the singular set of the free boundary of minimizers is empty if the dimension $n < k^*$. The authors in [10] showed $k^* > 3$ and also provided an example of a nontrivial stable solution in dimension $n = 7$. This example is analogous to the Simons cone and was later shown in [12] to indeed be a minimizer. Recently, the article [17] improved $k^* > 4$. It is still an open problem as to whether $k^* = 5, 6$, or 7 .

The article [21] further strengthened the connection between minimal surfaces and the free boundary problem by establishing a one-to-one correspondence between solutions of (1.1) in \mathbb{R}^2 and minimal bigraphs in \mathbb{R}^3 . The one-to-one correspondence further strengthens the principle of a reduction in one dimension when moving from the theory of minimal surfaces to the one phase problem. Recall for instance that $k^* = 8$ for area-minimizing surfaces where as k^* is most likely 7 for minimizers of (1.3).

1.2. Area-minimizing surfaces and the free boundary problem on cones.

In light of the connection described above, one may expect that results for the free boundary problem on cones are analogous to the results for area-minimizing surfaces on cones. The description of the free boundary problem on cones (2.3) as well as the corresponding functional (2.2) is given later in Section 2.

On two-dimensional cones area-minimizing surfaces of co-dimension 1 are distance minimizing geodesics. Two-dimensional cones in \mathbb{R}^3 are determined by the intersection of the cone with the two-sphere. If this intersection is a simple closed curve γ on S^2 , then the following Proposition is well known.

Proposition 1.1. *If $\text{length}(\gamma) < 2\pi$, no distance minimizing geodesics pass through the vertex. If $\text{length}(\gamma) \geq 2\pi$, then there are distance minimizing geodesics that pass through the vertex.*

The first statement in Proposition 1.1 can be found in Section 4-7 of [13]. The author with Chang Lara proved the following complete analogous result [2] for minimizers of (2.2) on a two-dimensional cone.

Theorem 1.2. *If u is a minimizer of (2.2) on a two-dimensional cone, and if $\text{length}(\gamma) < 2\pi$, then the vertex of the cone $0 \notin \partial\{u > 0\}$. If $\text{length}(\gamma) \geq 2\pi$, then the free boundary can pass through the vertex.*

The proof of Theorem 1.2 was the main result in [2] and utilized that two-dimensional cones are isometrically flat. A competitor with a smaller functional value was constructed via an iterative argument that depended on $\text{length}(\gamma)$.

In this paper we consider the free boundary problem on higher dimensional cones. In view of the connection between area-minimizing surfaces and the one phase free boundary problem, one is led to ask about area-minimizing surfaces on higher dimensional cones. Morgan [18] considered area-minimizing surfaces on n -dimensional cones defined by

$$(1.4) \quad x_{n+1} = c \sqrt{\sum_{i=1}^n x_i^2}$$

with $c \geq 0$, and proved that a k -dimensional plane through the vertex is area-minimizing if and only if

$$(1.5) \quad k \geq 3 \quad \text{and} \quad \delta^2 \geq \frac{4(k-1)}{k^2},$$

where $c = \delta^{-1}\sqrt{1-\delta^2}$. As a corollary (1.5) also determines when a k -dimensional area minimizing hypersurface can pass through a cone for $3 \leq k \leq 7$. In particular, a 3-dimensional area minimizing hypersurface may pass through the vertex of a 4-dimensional cone as given in (1.4) and this is determined by (1.5). By the aforementioned drop in one dimension from area-minimizing surfaces to the free boundary problem, one may expect that the lowest dimension for which a free boundary of a minimizer may pass through the vertex of a cone of type (1.4) is for a three dimensional cone, and this depends on the constant c . We prove this is indeed the case.

1.3. Main Results. We prove results analogous to those of area-minimizing surfaces on cones in [18]. In Section 3 we establish a second variational formula for minimizers of (2.2) on a cone of type (1.4). With a notion of second variation one may discuss whether a solution is stable. Our first result regards the stability of a homogeneous solution.

Theorem 1.3. *Let \mathcal{C} be a three-dimensional cone of type (1.4). There exists $0 < c_0 < \infty$ such that if $c \leq c_0$, then there exists a unique (up to rotation) 1-homogeneous solution of (2.3) that is stable. If $c > c_0$ then no 1-homogeneous solution of (2.3) is stable.*

From the above theorem we obtain the following

Corollary 1.4. *Let \mathcal{C} be a three-dimensional cone of type (1.4), and let c_0 be the constant in Theorem 1.3. If $c > c_0$ and u is a minimizer of (2.2), then the vertex $0 \notin \partial\{u > 0\}$.*

In the history of area-minimizing surfaces and free boundary problems it is common for stable solutions to indeed be global minimizers. Furthermore, the notion of stability and area-minimizing for hyperplanes on cones coincides [18]. Therefore, it is reasonable to assume that Theorem 1.3 may be improved by replacing the notion of stable with minimizer. In that vein we have our second main result.

Theorem 1.5. *Let \mathcal{C} be a three-dimensional cone of type (1.4). There exists c_1 with $0 < c_1 \leq c_0 < \infty$ such that if $c \leq c_1$, then there exists a minimizer u of (2.2) such that $0 \in \partial\{u > 0\}$.*

The significance of Theorem 1.3 is that $c_1 > 0$ which shows that three is the lowest dimension for which the free boundary of a minimizer passes through the vertex of a non-flat right circular cone. We expect that $c_1 = c_0$ in Theorem 1.5, but a proof that $c_1 = c_0$ may need to rely on numerical analysis such as in [12].

Many of the results in this paper apply to higher-dimensional cones of type (1.4). In Section 4 we present a symmetric 1-homogeneous solution Φ_c to (2.3) on \mathcal{C} where the vertex $0 \in \partial\{u > 0\}$. This symmetric solution Φ_c has a variant in each dimension. In Section 5 we show that if v is any homogeneous stable solution with $0 \in \partial\{v > 0\}$, and \mathcal{C} is three-dimensional, then $v \equiv \Phi_c$. Theorems 1.3 and 1.5 are reduced to showing whether the specific solution Φ_c is stable or is a minimizer. The symmetric candidate Φ_c is analogous to hyperplanes on cones as in [18]. As previously mentioned, there exist non-hyperplane cones that are area-minimizing; consequently, the results in [18] for hyperplanes can only be used to classify when an area-minimizing hypersurface of dimension k passes through the vertex of a cone when $k \leq 7$. Similarly, in order to prove Theorems 1.3 and 1.5 on higher-dimensional cones, one would have to show any stable homogeneous solution $v \equiv \Phi_c$. Although we only present results in this paper for the symmetric solution Φ_c on a three-dimensional cone, the same techniques will apply in higher dimensions. We state this in the following remark which is a partial analogue to Theorem 1.1 in [18] regarding hyperplanes on cones.

Remark 1.6. Let \mathcal{C} be a cone of type (1.4). Let Φ_c be the symmetric solution as defined in Section 4. If $n \geq 3$, then there exists two constants $0 < c_1 \leq c_0 < \infty$ depending on dimension n such that Φ_c is stable if and only if $c \leq c_0$ and Φ_c is a minimizer of (2.2) if $c \leq c_1$.

Finally, one may consider Lipschitz manifolds with isolated singularities. Suppose \mathcal{M} is a three-dimensional manifold and \mathcal{M} is smooth in a neighborhood $\Omega \setminus \{x_0\}$. If there exists a sequence $r_k \rightarrow 0$ such that a rescaling $\mathcal{M}_k \rightarrow \mathcal{C}$ where \mathcal{C} is of type (1.4), then the results of Theorem 1.5 will apply in a neighborhood of x_0 . This is because if the free boundary of a minimizer u passes through x_0 , then using

the regularity results in [14], one may obtain via a blow-up procedure a minimizer u_0 on \mathcal{C} and the vertex $0 \in \partial\{u_0 > 0\}$.

1.4. Outline and Notation. The outline of this paper is as follows. In Section 2 we define the notion of a solution to the free boundary problem and the corresponding functional. We also state some preliminary results necessary later in the paper. In Section 3 we give a second variation formula for 1-homogeneous solutions. In Section 4 we present a symmetric 1-homogeneous solution Φ_c . In Section 5 we classify when Φ_c is a stable solution and prove Theorem 1.3 and Corollary 1.4. In Section 6 we show that for c small but still positive the solution Φ_c is a minimizer of the functional (2.2) and thus prove Theorem 1.5.

We will use the following notation throughout the paper.

- n refers to dimension.
- \mathcal{C} is always a cone of type (1.4).
- c is always the constant appearing in the definition of a cone in (1.4).
- ∇_c refers to the gradient on \mathcal{C} arising from the inherited metric as explained in Section 2.
- Δ_c refers to the Laplace-Beltrami operator on \mathcal{C} as explained in 2.
- ∇_θ refers to the gradient on the sphere.
- Δ_θ represents the Laplace-Beltrami on the sphere.
- $\Gamma := \{u > 0\}$ where u is 1-homogeneous solution to (2.3).

2. PRELIMINARIES

We consider a 3-dimensional cone \mathcal{C} in \mathbb{R}^4 given by

$$(2.1) \quad \mathcal{C} := \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_4 = c\sqrt{y_1^2 + y_2^2 + y_3^2}\}.$$

We study minimizers of the functional

$$(2.2) \quad J(v, \Omega) := \int_{\Omega} |\nabla_c v|^2 + \chi_{\{v > 0\}},$$

where $\nabla_c v$ is the gradient on the cone \mathcal{C} away from the vertex. As shown later in Proposition 2.2, minimizers of (2.2) over $\Omega \subseteq \mathcal{C}$ are solutions to

$$(2.3) \quad \begin{cases} u \geq 0 \\ u \text{ is continuous in } \Omega \\ \Delta_c u = 0 \text{ in } \{u > 0\} \\ \partial\{u > 0\} \setminus \{0\} \text{ is locally smooth} \\ |\nabla_c u| = 1 \text{ on } \partial\{u > 0\} \setminus \{0\}, \end{cases}$$

where ∇_c is the gradient on \mathcal{C} from the inherited metric, and Δ_c is the Laplace-Beltrami on \mathcal{C} . The above class of solutions may seem restrictive; however, not only will all minimizers of (2.2) be solutions to (2.3), but also the common notion of viscosity solution as in [7–9] would also be a solution to (2.3) since the cone \mathcal{C} is three-dimensional.

We consider two main parametrizations of the cone \mathcal{C} . Using spherical coordinates we have

$$(2.4) \quad y_1 = r \cos \theta \sin \phi, \quad y_2 = r \sin \theta \sin \phi, \quad y_3 = r \cos \phi, \quad y_4 = cr.$$

Under these coordinates the area form is $\sqrt{\det(g)} = r^2 \sin \phi \sqrt{1+c^2}$. The local coordinates g^{ij} are given by

$$\begin{pmatrix} r^{-2} \sin^{-2} \phi & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (1+c^2)^{-1} \end{pmatrix},$$

and in these local coordinates we minimize

$$\int_{\Omega} \sqrt{g} g^{ij} u_i u_j + \sqrt{g} \chi_{\{u>0\}}.$$

Any minimizer will satisfy

$$\begin{cases} \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{ij} u_i) = 0 & \text{in } \{u > 0\} \\ \sqrt{g} g^{ij} u_i u_j = \sqrt{g} & \text{on } \partial\{u > 0\} \setminus \{0\}, \end{cases}$$

and so a minimizer of (2.2) is a solution to (2.3). We note that the first condition is written out as

$$(2.5) \quad \frac{1}{1+c^2} \left(u_{rr} + \frac{2}{r} u_r \right) + \frac{1}{r^2} \left(\frac{u_{\theta\theta}}{\sin \phi} + \frac{\cos \phi}{\sin \phi} u_{\phi} + u_{\phi\phi} \right) = 0$$

in the set $\{u > 0\}$.

We may also work in the coordinates

$$(2.6) \quad y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3, \quad y_4 = c \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

In these coordinates the area form is $\sqrt{g} = \sqrt{1+c^2}$ and the local coordinates g^{ij} are given by

$$\frac{1}{(1+c^2)r^2} \begin{pmatrix} r^2 + c^2(x_2^2 + x_3^2) & -c^2 x_1 x_2 & -c^2 x_1 x_3 \\ -c^2 x_1 x_2 & r^2 + c^2(x_1^2 + x_3^2) & -c^2 x_2 x_3 \\ -c^2 x_1 x_3 & -c^2 x_2 x_3 & r^2 + c^2(x_1^2 + x_2^2) \end{pmatrix}.$$

We define $\mathcal{C}_r := \{(y_1, y_2, y_3, y_4) \in \mathcal{C} : \sqrt{y_1^2 + y_2^2 + y_3^2} < r\}$. From the regularity results in [14], we have the following regarding the continuity of the minimizer as well as the growth away from a free boundary point at the vertex.

Proposition 2.1. *Let u be a minimizer of (2.2) on $\Omega \subseteq \mathcal{C}$, then u is Hölder continuous inside Ω . Furthermore, if the vertex $\{0\} \in \partial\{u > 0\}$, then there exists two constants C_1, C_2 depending on u such that*

$$C_1 r \leq \sup_{\mathcal{C}_r} u \leq C_2 r.$$

We also have

Proposition 2.2. *Let u be a minimizer of (2.2) in Ω . Then u is a solution to (2.3).*

Proof. From Proposition 2.1 minimizers are continuous. Then by considering variations in the positivity set, it follows that $\Delta_c u = 0$ in $\{u > 0\}$. Furthermore, the coefficients g^{ij} are smooth away from the vertex of the cone. Therefore, when the cone is three-dimensional one may combine the results in [15] and [10] to conclude that $\partial\{u > 0\}$ is smooth away from the vertex. It then follows from the domain variation techniques in [4] that the free boundary relation is satisfied away from the vertex so that any minimizer u of (2.2) is a solution to (2.3). \square

We also have the following Weiss-type monotonicity formula.

Proposition 2.3. *Let u be a minimizer to (2.2) on $\Omega \subseteq \mathcal{C}$ and $\mathcal{C}_R \subset \Omega$. Assume that the vertex $0 \in \partial\{u > 0\}$. Then the functional*

$$W(r, u) := \frac{1}{r^n} \int_{\mathcal{C}_r} |\nabla_c u|^2 + \chi_{\{u > 0\}} - \frac{1}{r^{n+1}} \int_{\partial \mathcal{C}_r} u^2$$

is monotone increasing in r for $r \leq R$. Furthermore, if $0 < r_1 < r_2 \leq R$ and $W(r_1, u) = W(r_2, u)$ if and only if u is homogeneous of degree 1 on $\mathcal{C}_{r_2} \setminus \mathcal{C}_{r_1}$.

The proof of Proposition 2.3 relies on a radial domain variation. Since \mathcal{C} may be parametrized by radial spherical coordinates, the usual proof will go through. For two-dimensional cones this was shown in [2]. The same proof has also been adapted for a more complicated weight (see [1, 3]). As a consequence of Propositions 2.1 and 2.3 we have the following

Proposition 2.4. *Let u be a minimizer of (2.2) on $\Omega \subseteq \mathcal{C}$ and assume the vertex $0 \in \partial\{u > 0\}$. Then there exists a sequence $r_k \rightarrow 0$ and a 1-homogeneous solution u_0 of (2.3) such that $0 \in \partial\{u_0 > 0\}$ and the rescaled functions $u_k := u(r_k x)/r_k$ converge uniformly on compact subsets of \mathcal{C} to u_0 . Furthermore, u_0 will be a minimizer of (2.2) on all $\Omega \Subset \mathcal{C}$.*

Proof. The following proof is standard. From Proposition 2.1, the Hölder estimates in [14], and the Aréla-Ascoli Theorem, there exists $r_k \rightarrow 0$ and u_0 such that $u_{r_k} \rightarrow u_0$ locally uniformly. That u_0 is a minimizer of (2.2) and therefore also a solution to (2.3) is standard. From the rescaling property $W(\rho r, u) = W(\rho, u_r)$ of the Weiss functional and the monotonicity of $W(r, u)$ it follows that $W(r, u_0) = W(0+, u)$ for all r , and so u_0 is homogeneous of degree 1. \square

3. A SECOND VARIATION FORMULA

In this section we adapt the ideas in [10] to obtain a second variation formula. From Proposition 2.4 we restrict ourselves to the study of 1-homogeneous solutions u to (2.3). We denote the positivity set of a 1-homogeneous solution u to (2.3) by $\Gamma := \{u > 0\}$. The free boundary $\partial\Gamma$ is a cone and we denote the mean curvature by H . The main result in this section is the following Lemma that gives a second variation formula for 1-homogeneous solutions to (2.3) that are also minimizers of (2.2). We define $\mathcal{F}_R := \{F \in C^\infty(\overline{\Gamma}) : F(x) = 0 \text{ if } |x| \notin (R^{-1}, R)\}$ and $\mathcal{F} := \cup \mathcal{F}_R$.

Lemma 3.1. *Let \mathcal{C} be a cone of type (2.1). Let u be a 1-homogeneous minimizer of (2.2). Let g^{ij} be the local coordinates from (2.6). Then*

$$(3.1) \quad \int_{\partial\Gamma} H F^2 d\sigma \leq \int_{\Gamma} g^{ij} F_i F_j \quad \text{for every } F \in \mathcal{F}$$

Proof. To prove Lemma 3.1 we intentionally follow the structure of the analogous Lemma in [10], so that the reader may compare.

We define $\Omega := \Gamma \cap B_R$ for some fixed R , and use the local coordinates g^{ij} as given in (2.6). We first assume that $F \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ solves $\partial_i(\sqrt{g} g^{ij} F_j) = 0$ in Γ . Since Γ is an NTA domain it is also a Twisted Hölder domain of order 1 so by Theorem 4.5 in [5] there is a boundary Harnack inequality for Δ_c on Γ . Then there exists a constant C such that $F \leq Cu$ in a neighborhood of the origin. We define

$$\Omega_\epsilon = \{x \in \Omega : u(x) > \epsilon F(x)\}.$$

Let $v_\epsilon = u - \epsilon F$ on $\overline{\Omega}_\epsilon$ and $v_\epsilon = 0$ on $B_R \setminus \Omega_\epsilon$. We then have that $v_\epsilon = u$ on ∂B_R . Integrating by parts we have

$$\int_{B_R \cap \{v_\epsilon > 0\}} g^{ij} \partial_i v_\epsilon \partial_j v_\epsilon = \int_{(\partial B_R) \cap \Gamma} u g^{ij} (u - \epsilon F)_i \nu_j.$$

Since $\sqrt{g} = \sqrt{1 + c^2}$ is a constant we may divide by \sqrt{g} when minimizing the functional. We then have

$$(3.2) \quad \frac{1}{\sqrt{1 + c^2}} (J(u, B_R) - J(v_\epsilon, B_R)) = \epsilon \int_{(\partial B_R) \cap \Gamma} u g^{ij} F_i \nu_j + \text{vol}(0 < u < \epsilon F).$$

We note that in the above equation the volume element is from the flat metric since we have already divided out by $\sqrt{1 + c^2}$. Integrating by parts on Ω we have

$$\begin{aligned} \int_{(\partial B_R) \cap \Gamma} u g^{ij} F_i \nu_j &= \int_{(\partial B_R) \cap \Gamma} (u g^{ij} F_i \nu_j - F g^{ij} u_i \nu_j) \\ &= \int_{\partial(B_R \cap \Gamma)} (u g^{ij} F_i \nu_j - F g^{ij} u_i \nu_j) - \int_{(\partial \Gamma) \cap B_R} F \\ &= - \int_{(\partial \Gamma) \cap B_R} F. \end{aligned}$$

Then we have that

$$\frac{1}{\sqrt{1 + c^2}} (J(u, B_R) - J(v_\epsilon, B_R)) = -\epsilon \int_{(\partial \Gamma) \cap B_R} f d\sigma + \text{vol}(0 < u < \epsilon F).$$

In [10] three remarks are given. The first remark is

Remark 3.2. $u_{\nu\nu} = -H$ on $\partial\Gamma$ except at the origin.

Proof. Under a rotation we assume our point $P \in \partial\Gamma$ to be $(x_1, 0, 0)$. Locally near P the free boundary is given by $u(x_1, x_2, \phi(x_1, x_2)) = 0$. Differentiating with respect to i, j (with $i, j = x_1, x_2$) we obtain

$$u_i + \phi_i u_{x_3} = 0; \quad u_{ij} + u_{ix_3} \phi_j + \phi_{ij} u_{x_3} + \phi_i u_{x_3 j} + \phi_j u_{x_3 i} = 0.$$

We now evaluate at P and use that $\phi_i(P) = 0$ and $u_{x_3}(P) = -1$ to conclude that

$$u_{ij}(P) = \phi_{ij}(P), \quad \text{for } i, j = x_1, x_2.$$

Recalling that H is the mean curvature of $\partial\Gamma$ at P we have that

$$(3.3) \quad u_{x_1 x_1} + u_{x_2 x_2} = H.$$

Now from the local coordinates given in (2.6) for $i, j = x_1, x_2, x_3$ we have

$$(\partial_j g^{ij}) u_i + g^{ij} u_{ij} = 0.$$

Evaluating at P and using that $u_{x_1}(P) = u_{x_2}(P)$ as well as $x_2 = x_3$ at P we obtain

$$\frac{1}{1 + c^2} u_{x_1 x_1}(P) + u_{x_2 x_2}(P) + u_{x_3 x_3}(P) = 0.$$

We finally note that because $\partial\Gamma$ is a cone, that $(t, 0, 0) \in \partial\Gamma$ for all $t \geq 0$, so that $u_{x_1 x_1}(P) = 0$. Then combining the above equation with (3.3) we obtain that

$$H = -u_{x_3 x_3}(P).$$

This concludes the proof of Remark 3.2. □

Remark 3.3. $H \geq 0$. In particular, $\mathbb{R}^3 \setminus \Gamma$ is a finite union of convex cones.

Whereas the proofs of Remarks 3.2 and 3.4 are very similar to those found in [10], the proof of Remark 3.3 is different and requires Lemma A.2 from the Appendix.

Proof of Remark 3.3. We utilize the homogeneity of u . Since $u = rf$ and $\Delta_c u = 0$ whenever $u > 0$ it follows that $\Delta r^\alpha f = 0$ as long as

$$\alpha = (-1 + \sqrt{1 + 8/(1 + c^2)})/2.$$

Notice that $0 < \alpha < 1$ for $c > 0$. From Lemma A.2, we have $|\nabla_\theta f|^2$ achieves the maximum on $S^2 \cap \partial\Gamma$. Since $|\nabla_\theta f| = 1$ everywhere on $\partial\Gamma$, we conclude that $|\nabla_\theta f| \leq 1$ in Γ . Under a rotation we may assume that $(1, 0, 0) = P \in \partial\Gamma$ and ϕ is the outward unit normal from the spherical coordinates given in (2.4). Then $\partial_\phi f_\phi^2 = 2f_\phi f_{\phi\phi} \geq 0$, and since $f_\phi(P) = -1$ we obtain that $f_{\phi\phi}(P) \leq 0$. Then $u_{33} \leq 0$, and it follows that $H \geq 0$. \square

Remark 3.4. Let $z : U \rightarrow \mathbb{R}^3$ defined on an open subset U of \mathbb{R}^2 be a local parametrization of the surface $\partial\Gamma$ and let $\nu = \nu(s)$ be the unit normal to $\partial\Gamma$ at $z(s)$ pointing away from Γ . In the coordinate system $x(s, t) = z(s) - t\nu(s)$, the volume element $dV = (1 + tH + O(t^2))d\sigma(s)dt$, where $d\sigma(s)$ is the $n - 1$ area element on the surface.

The volume element dV in Remark 3.4 is for the flat metric; therefore, Remark 3.4 is identical to that in [10] and the same proof applies.

We now finish the proof of Lemma 3.1. Note that in the local coordinates (2.6), if $P \in \partial\Gamma$, then $FF_\nu = g^{ij}F_i\nu_j$ at P where ν is the outward unit normal at P . This is most easily seen by under a rotation letting $P = (x_1, 0, 0)$. Now by combining Remarks 3.2, 3.3, and 3.4 as in [10] we obtain

$$\frac{1}{\sqrt{1 + c^2}}(J(u, B) - J(v_\epsilon, B)) = \epsilon^2 \int_{\partial\Gamma \cap B} (F^2 H - FF_\nu) d\sigma + O(\epsilon^3).$$

Since u is a minimizer of J we have that

$$\int_{\partial\Gamma \cap B} (F^2 H - FF_\nu) d\sigma \leq 0,$$

so that

$$\int_{\partial\Gamma \cap B} F^2 H d\sigma \leq \int_{\partial\Gamma \cap B} FF_\nu d\sigma = \int_{\partial\Gamma \cap B} g^{ij} F_i \nu_j d\sigma = \int_\Gamma g^{ij} F_i F_j d\sigma.$$

Since the above inequality is true for all $\Delta_c F = 0$ it follows that

$$\int_{\partial\Gamma \cap B} F^2 H d\sigma \leq \int_\Gamma g^{ij} F_i F_j d\sigma,$$

for any $F \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. This concludes the proof of Lemma 3.1. \square

We recall the following Lemma from [10].

Lemma 3.5. *Suppose that Γ is an open cone in \mathbb{R}^3 and $\partial\gamma = (\partial\Gamma) \cap S^2$ is a finite union of smooth curves. Then the mean curvature H of $\partial\Gamma$ can be written as*

$$H = \frac{1}{r} \kappa(x/r), \quad r = |x|$$

where κ is the geodesic curvature of the curve γ in the unit sphere S^2 .

Using Lemma 3.5 we may prove

Lemma 3.6. *Let Γ be a cone in \mathbb{R}^3 with the mean curvature of H of $\partial\Gamma$ satisfying $H \geq 0$ as well as (3.1) for every $F \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Then Γ^c has one connected component and is a convex cone contained in a half space.*

Proof. We choose F to be a radial function. We let $\gamma := \partial\Gamma \cap S^2$ and $U := S^2 \cap \Gamma$. Then

$$\int_\gamma \int_0^\infty \kappa(\alpha(s)) F^2(r) \, dr \, ds \leq \int_\Gamma g^{ij} F_i F_j,$$

where $\alpha(s)$ is a parametrization of γ with respect to arc length. We now convert the integral over Γ from the local coordinates given by (2.6) to the spherical coordinates given by (2.4).

$$\begin{aligned} \int_\Gamma g^{ij} F_i F_j &= \frac{1}{\sqrt{1+c^2}} \int_\Gamma \sqrt{g} g^{ij} F_i F_j \\ &= \frac{1}{\sqrt{1+c^2}} \int_c |\nabla_c F|^2 \\ &= \frac{1}{\sqrt{1+c^2}} \int_0^\infty \int_U r^2 \sin \phi \sqrt{1+c^2} (1+c^2)^{-1} F_r F \\ &= \frac{|U|}{1+c^2} \int_0^\infty r^2 F_r F, \end{aligned}$$

and $|U|$ is the area of U in the unit sphere. Using the same choice of radial function $F(|r|)$ as in [10], and combining the above two inequalities we obtain

$$\int_\gamma \kappa \, ds \leq \frac{1}{4(1+c^2)} |U|.$$

We now label V_j , $j = 1, \dots, m$ as the connected components of $S^2 \setminus U$ and γ_j as the boundary curves. Using that $H \geq 0$, so that $\kappa \geq 0$, we apply the Gauss-Bonnet formula

$$|V_j| + \int_{\gamma_j} \kappa \, ds = 2\pi,$$

and sum over j to obtain

$$2m\pi - \sum_{j=1}^m |V_j| = \sum_{j=1}^m \int_{\gamma_j} \kappa \, ds = \int_\gamma \kappa \, ds \leq \frac{1}{4(1+c^2)} |U|.$$

Since $\sum_{j=1}^m |V_j| = 4\pi - |U|$, we have that

$$0 < \left(1 - \frac{1}{4(1+c^2)}\right) |U| \leq 4\pi - 2m\pi.$$

Then $m = 1$, and U^c is a single connected convex component of S^2 , and so U^c must be contained in a half space. \square

In this Section we have closely followed the ideas in [10]. Moving forward, however, the ideas in this paper are very different. When $c = 0$ one may use a homogeneity argument to conclude that $\partial\Gamma$ is flat. When $c > 0$ we will see in the next Section that there exists a stable solution Φ_c where $\partial\Gamma$ is not flat. Moreover, in Section 6 we show that for $c > 0$ and small enough that these candidate solutions are indeed minimizers.

4. THE SYMMETRIC SOLUTION

In this Section we present a homogeneous solution Φ_c which will turn out to be stable and even a minimizer for certain values of c . We also show that up to rotation Φ_c is the only possible 1-homogeneous stable solution. From Lemma 3.5 if u is a 1-homogeneous solution that is stable, then $\{u = 0\}$ consists of a single connected component that is convex and contained in a half space. For each $c > 0$ we now describe a symmetric candidate solution. Using the spherical coordinates (2.4), if $\Delta_c r^\beta f = 0$ and $f(\theta, \phi)$ is a function of ϕ alone and independent of θ , then

$$(4.1) \quad \frac{\beta(\beta+1)}{1+c^2} f(\phi) + \frac{\cos \phi}{\sin \phi} f'(\phi) + f''(\phi) = 0.$$

Under the change of variables $t = \cos \phi$, the function f is a Legendre function and well understood. For $c \geq 0$ and fixed $\beta = 1$ there is a unique solution with $f'(\phi) = 0$ and $f'(\phi_0) = 1$ where $f(\phi_0) = 0$ which we will denote by $f_{1,c}$. We note that $f_{1,c}(\phi) \geq 0$ for $\phi \leq \pi/2$. Then $r f_{1,c}(\phi)$ is a symmetric (in the variable θ) candidate solution which we will denote by $\Phi_c(r, \phi) = r f_{1,c}(\phi)$. We now show that if u is a stable 1-homogeneous solution, then up to rotation $u \equiv \Phi_c$.

Lemma 4.1. *Let u be 1-homogeneous solution to (2.3) and assume that $\{u = 0\}$ is a single connected component contained in a half space. Then up to rotation $u \equiv \Phi_c$.*

Proof. We will use the moving plane method as presented in [19]. By rotation we will assume that $\{u = 0\}$ is contained in the hemisphere given by $0 \leq \phi \leq \pi/2$. Let $u = r g(\theta, \phi)$. We claim that for $0 \leq \phi \leq \pi/2$ we have $g(\theta, \phi) \leq g(\theta, \pi - \phi)$. That is when g is reflected across the equator the bottom half is always greater than or equal to the upper half. Let $h(\theta, \phi) = g(\theta, \pi - \phi)$ and consider $\{g > h\} \cap \{0 \leq \phi \leq \pi/2\}$. Notice that $(g - h)(\theta, \phi/2) = 0$ and is a nonnegative eigenfunction $\Delta_\theta(g - h) = -\lambda(g - h)$ on $\{g > h\} \cap \{0 \leq \phi \leq \pi/2\}$. Now $|\{g - h \leq 0\} \cap \{0 \leq \phi \leq \pi/2\}| > 0$, so that

$$\{g - h > 0\} \cap \{\phi \leq \pi/2\} \subset \{h > 0\} \quad \text{and} \quad |\{g - h > 0\} \cap \{\phi \leq \pi/2\}| < |\{h > 0\}|.$$

Both $g - h$ and h are both positive eigenfunctions on their respective positivity sets with the same eigenvalue. If $\{g - h > 0\} \cap \{\phi \leq \pi/2\} \neq \emptyset$, then the eigenvalue for $g - h$ would be strictly larger than the eigenvalue for h which would be a contradiction. Therefore, $g \leq h$ whenever $\phi \leq \pi/2$.

With this comparison principle in place one can begin to rotate the equator into the set $\{g = 0\}$. We have the same comparison argument as before, so that the reflection will always lie below. Then conditions (A), (B), (C), (D) of Section 3 in [19] are all met, so that we conclude $\{g = 0\}$ is a spherical cap. Now the nonnegative eigenfunction on a domain is unique, and after rotation there is a nonnegative eigenfunction given by (4.1). Since $u = r g$ and $\Delta_c u = 0$ it follows that $\beta = 1$ so that $g = f_{1,c}$, so that $u = \Phi_c$. □

5. STABLE SOLUTIONS

In this Section we prove Theorem 1.3. We first show that for c large enough the solution Φ_c is not stable.

Lemma 5.1. *There exists $M < \infty$ such that if $c \geq M$, then Φ_c is not a stable solution.*

Proof. Let ϕ_0 be such that $f_{1,c}(\phi_0) = 0$. We also label $t_0 = \cos \phi_0$. We have $t_0 \rightarrow -1$ as $c \rightarrow \infty$. Furthermore, the mean curvature H of $\partial\Gamma$ is H_1/r where r is the distance from the origin, and H_1 is the mean curvature at radius 1 which is

$$H_1 = \frac{|t_0|}{\sqrt{1-t_0^2}}.$$

We choose a radial function $F(r)$ which is smooth and compactly supported in $B_1 \setminus \{0\}$. Then

$$\begin{aligned} \int_{\partial\Gamma} H F^2 d\sigma &= \int_0^{|t_0|} H 2\pi \frac{1+t_0^2}{|t_0|} r F^2(t/t_0) \sqrt{1+(1-t_0^2)/t_0^2} dr \\ &= \int_0^{|t_0|} 2\pi F^2(t/t_0) |t_0|^{-1} dr \\ &= \int_0^1 F^2(r) dr. \end{aligned}$$

Thus, the above quantity remains constant independent of c . Furthermore, by converting the local coordinates from (2.6) to the spherical coordinates (2.4) we have

$$\begin{aligned} \int_{\Gamma} g^{ij} F_i F_j &= \frac{1}{\sqrt{1+c^2}} \int_{\mathcal{C}} |\nabla_c F|^2 \\ &= \frac{|S^2 \cap \{f_{1,c} > 0\}|}{1+c^2} \int_0^1 r^2 [F'(r)]^2 dr \\ &= \frac{2\pi[2/3 - |t_0| + |t_0|^3/3]}{1+c^2} \int_0^1 r^2 [f'(r)]^2 dr. \end{aligned}$$

As $c \rightarrow \infty$, the above quantity goes to zero. Therefore, for large enough c the second variational inequality (3.1) fails, and we conclude that our candidate solution Φ_c is not stable. \square

For this next Lemma we define $G_\beta(r, \phi) := r^\beta g_c(\phi)$ where $g_c(\phi)$ is a solution to (4.1) with $-1 < \beta < 0$ and the conditions $g'_c(0) = 0$ and $g_c(\phi) \geq 0$ for all $0 \leq \phi < \pi$. Now $G'_\beta \geq 0$ as well. We will see in the proof that it is convenient to choose $\beta = -1/2$ in which case we will simply write G . Also, when the value of c is fixed we will write simply $g(\phi)$. The function $g(\phi)$ should not be confused with the local coordinates g^{ij} from the metric defined by (2.6).

Lemma 5.2. *Let Φ_c be the symmetric solution as described in Section 4. Let $\Gamma = \{\Phi > 0\}$, and ϕ_0 such that $\Phi_c(\phi_0) = 0$. Then (3.1) is equivalent to*

$$(5.1) \quad H_1 \leq \frac{g'(\phi_0)}{g(\phi_0)}$$

where H_1 is the mean curvature of Γ at $r = 1$, and $g(\phi)$ is a nonnegative solution to (4.1) with

$$\beta = -1/2, \quad g'(0) = 0, \quad g(\phi) > 0, \quad g'(\phi) \geq 0.$$

Proof. We first show that (5.1) implies (3.1). Let $F \in \mathcal{F}$ with $F = 0$ on $\partial B_{R_2} \cap \Gamma$ and $\partial B_{R_1} \cap \Gamma$. We aim to minimize the quantity

$$E(F, \Gamma) := \frac{\int_{\Gamma} g^{ij} F_i F_j}{\int_{\partial \Gamma} H F^2 d\sigma}$$

From the standard theory of Calculus of variations, a unique minimizer v will exist and satisfy the Steklov eigenvalue problem

$$\begin{cases} \partial_j(g^{ij}v_i) = 0 & \text{in } \Gamma \cap (B_{R_2} \setminus B_{R_1}) \\ g^{ij}v_i\nu_j = \lambda H v & \text{on } \partial \Gamma \cap (B_{R_2} \setminus B_{R_1}), \end{cases}$$

where $\lambda = E(v, \Gamma)$. In order to obtain a lower bound for λ , we consider the functions G_β . Since $G_\beta > 0$ on $\partial \Gamma \cap (B_{R_2} \setminus B_{R_1})$, there exists some constant M and a point $x_0 \in \Gamma \cap (B_{R_2} \setminus B_{R_1})$ such that $M G_\beta \geq v$ and $M G_\beta(x_0) = v(x_0)$. Since $\Delta_c G_\beta = 0$ in Γ , then from the maximum principle we conclude $x_0 \in \partial \Gamma \cap (B_{R_2} \setminus B_{R_1})$. Then if $x_0 = (r_0, \theta_0, \phi_0)$ and H_1 is the mean curvature of Γ at $x_0/|r|$ we obtain

$$\begin{aligned} r^\beta \frac{M g'(\phi_0)}{r} &= g^{ij} \partial_i M G_\beta \nu_j(x_0) \\ &\leq g^{ij}(x_0) v_i \nu_j \\ &= \lambda H v(x_0) \\ &= r^\beta \lambda H_1 \frac{M g(x_0)}{r}. \end{aligned}$$

Thus

$$\frac{g'(\phi_0)}{g(\phi_0)} \leq \lambda H_1 = E(v, \Gamma) H_1.$$

Then if

$$1 \leq \frac{1}{H_1} \frac{g'(\phi_0)}{g(\phi_0)},$$

then also $E(v, \Gamma) \geq 1$, and so (3.1) holds.

We now show that (3.1) implies (5.1). From the symmetry of β for $-1 < \beta < 0$ in (4.1), one may expect a maximum bound from below when $\beta = -1/2$. We now show this is indeed the case. Recall that $G(r, \phi) = r^{-1/2} g(\phi)$ where $g(\phi)$ is a solution to (4.1) with $\beta = -1/2$. Then in local coordinates (2.6) we have

$$\begin{aligned} \int_{\Gamma \cap (B_{R_2} \setminus B_{R_1})} g^{ij} G_i G_j &= \int_{\partial \Gamma} g^{ij} G_i \nu_j + \int_{\partial B_{R_2} \cap \Gamma} g^{ij} G_i \nu_j + \int_{\partial B_{R_1} \cap \Gamma} g^{ij} G_i \nu_j \\ &= \int_{\partial \Gamma} g^{ij} G_i \nu_j, \end{aligned}$$

since

$$\int_{\partial B_{R_2} \cap \Gamma} g^{ij} G G_i \nu_j + \int_{\partial B_{R_1} \cap \Gamma} g^{ij} G_i \nu_j = 0,$$

because $G(r, \phi) = r^{-1/2} g(\phi)$. This is also why we chose $\beta = -1/2$. Now

$$\int_{\partial \Gamma} g^{ij} G G_i \nu_j = \int_{\partial \Gamma} r^{-1/2} g(\phi_0) r^{-1/2} g'(\phi_0) = \int_{\partial \Gamma} r^{-1} g(\phi_0) g'(\phi_0).$$

We now define

$$\Psi_1 := g(\phi)[2R_1^{-3/2}(r - R_1/2)]_+,$$

and note that

$$\int_{\Gamma \cap B_{R_1}} g^{ij} \partial_i \Psi_1 \partial_j \Psi_1 \leq C_1 \int_0^{R_1} r^2 R_1^{-3} \leq C_1.$$

Similarly if

$$\Psi_2 := -g(\phi) R_2^{-3/2} [r - 2R]_+,$$

then

$$\int_{\Gamma \setminus B_{R_2}} g^{ij} \partial_i \Psi_2 \partial_j \Psi_2 \leq C_2 \int_0^{2R_2} r^2 R_2^{-3} \leq C_2.$$

Now if we define

$$\tilde{F}(x) := \begin{cases} \Psi_1(x) & \text{if } r < R_1 \\ G(x) & \text{if } R_1 \leq r \leq R_2 \\ \Psi_2(x) & \text{if } r > R_2, \end{cases}$$

then

$$\begin{aligned} \frac{1}{H_1} \frac{g'(\phi_0)}{g(\phi_0)} &\leq E(\tilde{F}, \Gamma) \\ &\leq \frac{\int_{\Gamma} g^{ij} \tilde{F}_i \tilde{F}_j}{\int_{\partial \Gamma \cap (B_{R_2} \setminus B_{R_1})} H r^{-1/2} g^2(\phi_0)} \\ &\leq \frac{C_1 + C_2}{\int_{\partial \Gamma \cap (B_{R_2} \setminus B_{R_1})} H r^{-1/2} g^2(\phi_0)} + \frac{\int_{\Gamma} g^{ij} \tilde{F}_i \tilde{F}_j}{\int_{\partial \Gamma \cap (B_{R_2} \setminus B_{R_1})} H r^{-1/2} g^2(\phi_0)} \\ &= \frac{C_1 + C_2}{\int_{\partial \Gamma \cap (B_{R_2} \setminus B_{R_1})} H r^{-1/2} g^2(\phi_0)} + \frac{g'(\phi_0)g(\phi_0)}{H_1 g^2 \phi_0}. \end{aligned}$$

Now as $R_1 \rightarrow 0$ and $R_2 \rightarrow \infty$ the first term in the last line above goes to zero. Hence, we conclude that

$$\min E(F, \Gamma) = \frac{1}{H_1} \frac{g'(\phi_0)}{g(\phi_0)}.$$

Then (3.1) holds if and only if $H_1 \leq g'(\phi_0)/g(\phi_0)$. \square

Corollary 5.3. *There exists $\epsilon_0 > 0$ such that if $c \leq \epsilon_0$, then Φ_c is a stable solution; i.e., if $\Gamma = \{\Phi_c > 0\}$, then (3.1) holds.*

Proof. From Lemma (5.1) we have that (3.1) holds if and only if

$$H_1 \leq \frac{g'_c(\phi_0)}{g_c(\phi_0)}.$$

As $c \rightarrow 0$ we have that $H_1 \rightarrow 0$, $\pi/2 \leq \phi_0 \rightarrow \pi/2$, and $g_c \rightarrow \cos \phi$. Then the above inequality will be satisfied for small enough c . \square

We are now ready to prove our first Main Theorem.

Proof of Theorem 1.3. We first normalize by letting $g_c(0) = 1$. If $0 \leq c_1 < c_2$, the g_{c_1} is a supersolution to (4.1) for c_2 , and so $g_{c_1} > g_{c_2}$ on $(0, \pi)$. Furthermore, for fixed ϕ_0 if $g_{c_1}(\phi_0) - h = g_{c_2}(\phi_0)$, then $g_{c_2} - h$ is a subsolution on the interval (ϕ_0, π) to (4.1) for c_2 . Then $g_{c_1} - h > g_{c_2}$ on (ϕ_0, π) . It follows that $g'_{c_1} > g'_{c_2}$

on $(0, \pi)$. Then g_{c_1}/g_{c_2} is increasing on $(0, \pi)$, so that $\log(g_{c_2}/g_{c_1})$ is increasing on $(0, \pi)$. Then

$$(5.2) \quad \frac{g'_{c_1}}{g_{c_1}} \geq \frac{g'_{c_2}}{g_{c_2}} \text{ on } (0, \pi) \text{ if } 0 \leq c_1 < c_2.$$

Now if $g_{c_2}(\phi_2) = 0$, then the mean curvature of Γ is

$$H_1 = -\frac{\cos \phi_2}{\sin \phi_2}.$$

Then (3.1) which is equivalent to (5.1) holds if and only if

$$(5.3) \quad 1 \leq -\frac{\sin \phi_2}{\cos \phi_2} \frac{g'_{c_2}(\phi_2)}{g_{c_2}(\phi_2)}.$$

From Corollary 5.3 there exists $c_2 > 0$ such that Φ_{c_2} is a stable solution, so that (5.3) holds where $g_{c_2}(\phi_2) = 0$. We seek to show that if $0 \leq c_1 < c_2$, then (5.3) holds for c_1 where $g_{c_1}(\phi_1) = 0$. We take the derivative

$$\begin{aligned} \frac{d}{d\phi} \left(-\frac{\sin \phi}{\cos \phi} \frac{g'_{c_2}(\phi)}{g_{c_2}(\phi)} \right) &= -\frac{1}{\cos^2 \phi} \frac{g'_{c_2}(\phi)}{g_{c_2}(\phi)} \\ &\quad - \frac{\sin \phi}{\cos \phi} \left[\frac{g_{c_2}(\phi)g''_{c_2}(\phi) - [g'_{c_2}(\phi)]^2}{g_{c_2}^2(\phi)} \right] \\ &= -\frac{1}{\cos^2 \phi} \frac{g'_{c_2}(\phi)}{g_{c_2}(\phi)} \\ &\quad + \frac{g'_{c_2}(\phi)}{g_{c_2}(\phi)} - \frac{\sin \phi}{\cos \phi} \frac{1}{4(1+c^2)} + \frac{\sin \phi}{\cos \phi} \frac{[g'_{c_2}(\phi)]^2}{g_{c_2}^2(\phi)} \\ &= \frac{\sin \phi}{\cos \phi} \left[-\frac{\sin \phi}{\cos \phi} \frac{g'_{c_2}(\phi)}{g_{c_2}(\phi)} + \frac{[g'_{c_2}(\phi)]^2}{g_{c_2}^2(\phi)} - \frac{1}{4(1+c^2)} \right]. \end{aligned}$$

Since (5.3) holds at ϕ_2 , and since $\phi_2 > \pi/2$, it follows that in a small neighborhood around $(\phi_2 - \epsilon, \phi_2 + \epsilon)$ that the above derivative is negative. Then if $c_3 \in (c_2 - \delta, c_2 + \delta)$ for small enough delta, then $\phi_3 \in (\phi_2 - \epsilon, \phi_2 + \epsilon)$ where $g_{c_3}(\phi_3) = 0$. Then if $c_3 \in (c_2 - \delta, c_2)$ we have that

$$1 \leq -\frac{\sin \phi_2}{\cos \phi_2} \frac{g'_{c_2}(\phi_2)}{g_{c_2}(\phi_2)} \leq -\frac{\sin \phi_3}{\cos \phi_3} \frac{g'_{c_2}(\phi_3)}{g_{c_2}(\phi_3)} \leq -\frac{\sin \phi_3}{\cos \phi_3} \frac{g'_{c_3}(\phi_3)}{g_{c_3}(\phi_3)}.$$

The last inequality follows from (5.2). We have shown that the set of points $c \in [0, \infty)$ for which (5.3) holds is open to the left. Since the inequality is preserved in a limit, it follows that the set of points $c \in [0, \infty)$ for which (5.3) holds is also closed to the left. Combining this with Lemma 5.1 there is then a last point $c_0 < \infty$ such that Φ_c is stable if and only if $0 \leq c \leq c_0$. From Corollary 5.3 we have that $c_0 > 0$. This concludes the proof. \square

We now give the

Proof of Corollary 1.4. Let u be a minimizer of (2.2) with $c > c_0$ with c_0 given in Theorem 1.3. Suppose by way of contradiction that the vertex $0 \in \partial\{u > 0\}$. By Proposition 2.4 there exists a 1-homogeneous minimizer u_0 of (2.2). Then u_0 is also stable, and so by Lemmas 3.6 and 4.1 we conclude $u_0 \equiv \Phi_c$. But Φ_c is not stable for $c > c_0$, and we obtain a contradiction. \square

6. A MINIMIZER FOR $c > 0$.

In this section we prove that the symmetric solution Φ_c defined in Section 4 is indeed a minimizer for $0 \leq c \leq c_0$ for c_0 small enough. This is accomplished by trapping Φ_c between a continuous family of sub- and supersolutions to the free boundary problem (2.3). This shows that Φ_c is a unique solution subject to its own boundary data. Since a minimizer does exist and is a solution, then Φ_c is a minimizer. We first construct a continuous family of subsolutions from below.

Lemma 6.1. *There exists $c_0 > 0$ such that if $0 \leq c \leq c_0$, and if u is a solution to (2.3) with $u = \Phi_c$ on ∂B_1 , then $u \geq \Phi_c$ in B_1 .*

Proof. We let $\Phi_c = r f_{1,c}(\phi)$. For convenience throughout this proof we will simply write $f(\phi)$ in place of $f_{1,c}(\phi)$. We consider $(v_\epsilon)_+$ where $v_\epsilon := r f(\phi) - \epsilon r^\beta g(\phi)$ and

$$g(\phi) := M - \cos(\phi).$$

Notice that

$$\Delta_c r^\beta g(\phi) = r^{\beta-2} \left[\frac{\beta(\beta+1)}{1+c^2} (M - \cos(\phi)) + 2 \cos(\phi) \right]$$

By choosing $-1 < \beta < 0$ and M large enough depending on β and c_0 , then $\Delta_c r^\beta g(\phi) \leq 0$ for $0 \leq c \leq c_0$. Thus, $\Delta_c (v_\epsilon)_+ \geq 0$ independent of ϵ . For convenience throughout the remainder of the proof we will write simply v in place of v_ϵ .

We note that

$$\begin{aligned} v_r &= f(\phi) - \epsilon \beta r^{\beta-1} g(\phi). \\ v_\phi &= r f'(\phi) - \epsilon r^\beta g'(\phi). \end{aligned}$$

Furthermore, on $\{v = 0\}$ we have $r f(\phi) = \epsilon r^\beta g(\phi)$, so that on $\{v = 0\}$ we obtain

$$(6.1) \quad |\nabla_c v|^2 = \frac{1}{1+c^2} v_r^2 + \frac{1}{r^2} v_\phi^2 = \frac{(1-\beta)^2}{1+c^2} f^2(\phi) + \left[f'(\phi) - f(\phi) \frac{g'(\phi)}{g(\phi)} \right]^2.$$

In order to use a comparison principle, we need $|\nabla_c v|^2 > 1$ on $\partial\{v > 0\}$. We let ϕ_0 be such that $f(\phi_0) = 0$. Notice that $\phi_0 > \pi/2$. Furthermore, since $\epsilon r^\beta g(\phi) \geq 0$, then $v(r, \phi) = 0$ only when $\phi < \phi_0$. Finally, we note that

$$(6.2) \quad |\nabla_c v(r, \phi_0)|^2 = \left[f'(\phi_0) - f(\phi_0) \frac{g'(\phi_0)}{g(\phi_0)} \right]^2 = |f'(\phi_0)|^2 = |\nabla_c \Phi_c(r, \phi_0)|^2 = 1.$$

We now take the derivative in ϕ of the last expression in (6.1).

$$\begin{aligned} & \frac{d}{d\phi} \left(\frac{(1-\beta)^2}{1+c^2} f^2(\phi) + \left[f'(\phi) - f(\phi) \frac{g'(\phi)}{g(\phi)} \right]^2 \right) \\ &= \frac{2(1-\beta)^2}{1+c^2} f(\phi) f'(\phi) + 2 \left[f'(\phi) - f(\phi) \frac{g'(\phi)}{g(\phi)} \right] \times \\ & \quad \left[f''(\phi) - f'(\phi) \frac{g'(\phi)}{g(\phi)} - f(\phi) \left(\frac{g(\phi) g''(\phi) - [g'(\phi)]^2}{g^2(\phi)} \right) \right]. \end{aligned}$$

We recall that

$$f''(\phi) = -\frac{\cos \phi}{\sin \phi} f'(\phi) - \frac{2}{1+c^2} f(\phi).$$

Substituting this into the computed derivative, reorganizing terms, and dividing by 2, we obtain that the derivative (divided by 2) is the sum of the following three pieces

$$\begin{aligned}
 (6.3) \quad & \left[\frac{(1-\beta)^2}{1+c^2} - \frac{2}{1+c^2} - \left(\frac{g(\phi)g''(\phi) - [g'(\phi)]^2}{g^2(\phi)} \right) \right] f(\phi)f'(\phi) \\
 & + f^2(\phi) \frac{g'(\phi)}{g(\phi)} \left[\frac{2}{1+c^2} + \left(\frac{g(\phi)g''(\phi) - [g'(\phi)]^2}{g^2(\phi)} \right) \right] \\
 & + \left[f'(\phi) - f(\phi) \frac{g'(\phi)}{g(\phi)} \right] \left[\frac{\cos \phi}{\sin \phi} + \frac{g'(\phi)}{g(\phi)} \right] (-f'(\phi)) \\
 & = I + II + III.
 \end{aligned}$$

We choose $\beta = -1/2$. We also have that $f''(\phi) < -\delta_1 < 0$ for $0 \leq \phi \leq \phi_0$ and some $\delta_1 > 0$ depending on c_0 and independent of c if $0 \leq c \leq c_0$. Then choosing M large enough depending on c_0 , there exists a constant δ_2 depending on c_0 such that for $0 \leq c \leq c_0$ we have

$$f(\phi) \frac{g'(\phi)}{g(\phi)} \leq \delta_2 f'(\phi) \text{ for } \phi \geq \phi_0.$$

Then

$$I + II \leq \left[\frac{(1-\beta)^2}{1+c^2} - \frac{2+\delta_2}{1+c^2} - (1+\delta_2) \left(\frac{g(\phi)g''(\phi) - [g'(\phi)]^2}{g^2(\phi)} \right) \right] f(\phi)f'(\phi).$$

Choosing again M large enough, there exists C_1 such that if $0 \leq c \leq c_0$, then

$$I + II \leq C_1 f(\phi)f'(\phi) \leq 0 \quad \text{for } 0 \leq \phi \leq \phi_0.$$

The parameter M is now fixed. For c_0 small, the angle ϕ_0 is close to $\pi/2$. Therefore, to control III we choose c_0 small enough so that

$$\frac{\cos \phi}{\sin \phi} + \frac{g'(\phi)}{g(\phi)} > 0 \quad \text{for } \phi \leq \phi_0.$$

Then $III \leq 0$, and so $I + II + III \leq 0$. Furthermore, in the above proof it is clear that $I + II + III < 0$ when $0 < \phi < \phi_0$, so that $|\nabla_c v|^2 > 1$ on $\partial\{v = 0\}$. Thus we have shown that $\Delta(v_\epsilon)_+ \geq 0$ and $|\nabla(v_\epsilon)_+| > 1$ on $\partial\{(v_\epsilon)_+ > 0\}$, and this is independent of $\epsilon > 0$.

Now let u be a solution to (2.3) with $u = \Phi_c$ on ∂B_1 . Suppose by way of contradiction that there exists $x_0 \in B_1$ such that $u(x_0) < \Phi_c(x_0)$. We have that $v_\epsilon < u$ on ∂B_1 for all $\epsilon > 0$, and we may choose ϵ large enough so that $v_\epsilon < u$ in B_1 . Then $(v_\epsilon)_+ < u$ in $\{u > 0\}$. Also, $\{(v_\epsilon)_+ > 0\} \subset \{u > 0\}$. We now continuously shrink ϵ until either $(v_\epsilon)_+$ touches u from below in $\{u > 0\}$, or $\partial\{(v_\epsilon)_+ > 0\}$ touches $\partial\{u > 0\}$. Since $(v_\epsilon)_+ \rightarrow \Phi_c$ pointwise on $B_1 \setminus \{0\}$, there exists an ϵ_0 and $x_1 \neq 0$ such that $(v_{\epsilon_0})_+ \leq u$ in B_1 and either $(v_{\epsilon_0})_+(x_1) = u(x_1) > 0$ or $x_1 \in (\partial\{(v_{\epsilon_0})_+ > 0\} \cap \partial\{u > 0\})$. Since $\Delta_c(v_{\epsilon_0}) > 0$ in $\{(v_{\epsilon_0})_+ > 0\}$, the first possibility is a violation of the comparison principle. Since $(v_{\epsilon_0})_+ \leq u$, if $(v_{\epsilon_0})_+(x_1) = u(x_1) = 0$, then since $|\nabla_c u(x_1)| = 1 < |\nabla_c(v_{\epsilon_0})_+|$ we also obtain a contradiction. Therefore, if $0 \leq c \leq c_0$ and if u is a solution to (2.3) with $u = \Phi_c$ on ∂B_1 , then $u \geq \Phi_c$. \square

Lemma 6.2. *There exists $c_0 > 0$ such that if $0 \leq c \leq c_0$, and if u is a solution to (2.3) with $u = \Phi_c$ on ∂B_1 , then $u \leq \Phi_c$.*

Proof. The beginning of the proof is similar to the proof of Lemma 6.1. We consider $v_c = rf_{1,c}(\phi) + \epsilon r^\beta g(\phi)$ with $\beta = -1/2$ and $g(\phi) = M - \cos \phi$. We will write f in place of $f_{1,c}$ when c is understood. We use the subscript c on the function v_c because later in the proof we will let c vary.

On the set $\{v_c = 0\}$ we have $rf(\phi) = -r^\beta g(\phi)$, so that once again we obtain that on $\{v_c = 0\}$ we have

$$|\nabla_c v_c|^2 = \frac{(1-\beta)^2}{1+c^2} f^2(\phi) + \left[f'(\phi) - f(\phi) \frac{g'(\phi)}{g(\phi)} \right]^2.$$

One main difference from the proof of Lemma 6.1 is that now if $v_c(r, \phi) = 0$, then $\phi \geq \phi_0$ where $f(\phi_0) = 0$. Using the same computations as in the proof of Lemma 6.1, we obtain by taking the derivative in ϕ that $|\nabla_c v_c(r, \phi)|^2 < 1$ provided that $\phi < \phi_1$ where $\phi_1 > \phi_0$ and ϕ_1 is determined by letting c be small so that the third term *III* in (6.3) is negative. We now fix ϕ_2 with $\phi_0 < \phi_2 < \phi_1$. Notice that ϕ_0 depends on c , but for small enough c_0 , ϕ_1 and ϕ_2 will not depend on c for $0 \leq c \leq c_0$. For fixed ϵ_0 , let r_0 be such that

$$r_0 f(\phi_2) + \epsilon_0 r_0^\beta g(\phi_2) = 0.$$

Then

$$(6.4) \quad \epsilon_0 = -r_0^{1-\beta} \frac{f(\phi_2)}{g(\phi_2)}.$$

We have that $|\nabla_c v_c| > 1$ on $\partial\{v > 0\}$ as long as $\phi \leq \phi_2$. We now redefine the function v_c on B_{r_0} . We first notice that from (6.4), we may rescale by

$$v_{r_0} := \frac{v_c(r_0 x)}{r_0}.$$

The rescaled function v_{r_0} is defined on B_{1/r_0} . Therefore, we may assume without loss of generality, that $\epsilon = -f(\phi_2)/g(\phi_2)$ and that we are redefining the values on B_1 . We now define $U := B_1 \cap \{x_3 > \cos(\phi_2)\}$ and also define

$$\tilde{v}_c := \begin{cases} \Delta_c \tilde{v}_c = 0 & \text{in } U \\ \tilde{v}_c = v & \text{on } \partial U \cap \{x_3 > \cos(\phi_2)\} \\ \tilde{v}_c = 0 & \text{on } \partial U \cap \{x_3 = \cos(\phi_2)\}. \end{cases}$$

Finally, we paste the two functions \tilde{v}_c and v_c by defining

$$(6.5) \quad w_c := \begin{cases} (v_c)_+ & \text{in } B_1^c \\ \tilde{v}_c & \text{in } U \\ 0 & \text{in } B_1 \setminus U. \end{cases}$$

Using a compactness argument, we will show for ϕ_2 fixed and small enough c , that $\Delta_c w_c \leq 0$ in $\{w_c > 0\}$ and $|\nabla_c w_c| < 1$ on $\partial\{w_c > 0\} \cap B_1^c$ and $\partial\{w_c > 0\} \cap B_1$. Now $\{w_c > 0\}$ is a wedge-type domain at $\partial\{w_c > 0\} \cap \partial B_1$, but because of the angle of the wedge we will see that $\partial\{w_c > 0\} \cap \partial B_1$ can never touch the free boundary $\partial\{u > 0\}$ of a solution u to (2.3).

We first show that if $c = 0$, we obtain the needed properties for w_0 . Then using a compactness argument, we show that if c is small enough, that w_c will also have the needed properties. If $c = 0$, then on ∂B_1 and $\phi < \phi_2$ we have that

$$v_0(1, \phi) = \cos \phi - \frac{\cos \phi_2}{g(\phi_2)} g(\phi),$$

so that

$$\tilde{v}_0 = r \left[\cos \phi - \frac{\cos(\phi_2)}{M - \cos \phi_2} (M - \cos \phi) \right].$$

This is just a linear function, and we notice that

$$\frac{\partial \tilde{v}_0}{\partial x_3}(x_1, x_2, -\cos \phi_2) = 1 + \frac{\cos(\phi_2)}{M - \cos \phi_2} < 1.$$

Furthermore, on ∂B_1 with $x_3 > \cos \phi_2$, if ν is the outward unit normal to B_1 we have

$$\begin{aligned} \frac{d\tilde{v}_c}{d\nu}(1, \phi) &= \cos \phi - \frac{\cos \phi_2}{M - \cos \phi_2} (M - \cos \phi) \\ &> \cos \phi - \beta \frac{\cos \phi_2}{M - \cos \phi_2} \cos \phi \\ &= \frac{d(v_c)_+}{d\nu}(1, \phi), \end{aligned}$$

as long as $M > 2$. Then $\Delta w_0 > 0$ weakly in $\{w_0 > 0\}$.

Now U is not a C^1 domain. However by Lemma B.3, if we let $c \rightarrow 0$, then $\tilde{v}_c \rightarrow \tilde{v}_0$ in C^1 on \overline{U} . Thus, for c small enough we obtain that

$$|\nabla_c \tilde{v}_c| < 1 \text{ on } \{x_3 = \cos \phi_2\} \cap B_1$$

and

$$\frac{d\tilde{v}_c}{d\nu}(1, \phi) - \frac{d(v_c)_+}{d\nu}(1, \phi) > 0.$$

Thus, for c_0 small enough and for $0 \leq c \leq c_0$, we have $\Delta_c w_c < 0$ weakly in $\{w_c > 0\}$, and a comparison principle holds.

We now let c_0 be chosen as above. Let $0 \leq c \leq c_0$ and let u be a solution to (2.3) with $u = \Phi_c$ on ∂B_ρ with $\rho < -\cos \phi_2$ where ϕ_2 defines U . Suppose that there exists $x_0 \in B_\rho$ such that $u(x) > \Phi_c(x)$. We may choose ϵ large enough so that $w_\epsilon > u$ on \overline{B}_ρ . Since $w_\epsilon > \Phi_c$ in B_1 for every $\epsilon > 0$, then also $w_\epsilon > u$ on ∂B_ρ for every $\epsilon > 0$. By continuously moving ϵ towards 0, there exists ϵ_1 and $x_1 \neq 0$ such that $w_{\epsilon_1} \geq u$ and either $w_{\epsilon_1}(x_1) = u(x_1) > 0$ or $x_1 \in \partial\{w_{\epsilon_1} > 0\} \cap \{u > 0\}$. If $w_{\epsilon_1}(x_1) = u(x_1) > 0$, we obtain a contradiction since $\Delta_c w_{\epsilon_1} < 0$ weakly in $\{w_{\epsilon_1} > 0\}$. If $x_1 \in \partial\{w_{\epsilon_1} > 0\} \cap \{u > 0\}$ and $|x_1| \neq 1$, we again obtain a contradiction since $w_{\epsilon_1} \geq u$, $|\nabla w_{\epsilon_1}(x_1)| < 1$, and $|\nabla u(x_1)| = 1$. We now consider the last case in which $|x_1| = 1$ and $x_1 \in \partial\{w_{\epsilon_1} > 0\} \cap \{u > 0\}$. Since $|\nabla v_{\epsilon_1}| < 1$ on $\partial B_1 \cap \partial\{v_{\epsilon_1} > 0\}$ and $|\nabla \tilde{v}_{\epsilon_1}| < 1$ on $\partial B_1 \cap \{\tilde{v}_{\epsilon_1} > 0\}$. Then

$$\sup_{B_t(x_1)} w_{\epsilon_1} \leq \delta_3 t,$$

for $x_1 \in \partial B_1 \cap \partial\{w_{\epsilon_1} > 0\}$ and for some constant $0 < \delta_3 < 1$ and t small enough. We then again obtain a contradiction since $w_{\epsilon_1} \geq u$. Therefore, $u \leq \Phi_c$ on B_ρ . Since Φ_c is homogeneous, then by rescaling it is also true that $u \leq \Phi_c$ for any solution u to (2.3) with $u = \Phi_c$ on ∂B_1 . \square

We now give the proof of our second main Theorem.

Proof of Theorem 1.5. Let $0 \leq c \leq c_0$. From the Calculus of Variations there exists a minimizer u of (2.2) with $u = \Phi_c$ on ∂B_1 . By Proposition 2.2 the minimizer u is a solution to (2.3) in B_1 . By Lemma 6.2 we have $u \leq \Phi_c$. The c_0 in the statement of Lemma 6.1 is greater than or equal to the c_0 in the statement of Lemma 6.2, so

that from Lemma 6.1 we also have $u \geq \Phi_c$ in B_1 . Then $u \equiv \Phi_c$, and therefore Φ_c is a minimizer. \square

APPENDIX A. A MAXIMUM PRINCIPLE

In order to prove nonnegative mean curvature of the free boundary of a homogeneous solution, we will need two Lemmas. If $v = rf(\theta)$ so that v is homogeneous of degree 1, then

$$|\nabla v|^2 = f^2 + |\nabla_\theta f|^2.$$

Consequently,

$$v^2(x) \leq |\nabla v(x)|^2 \quad \text{for any } x \in S^{n-1}.$$

If u is homogeneous of degree 0, we have a similar result for the Hessian and gradient. Although the following Lemma is not difficult to show in all dimensions via induction, we only state and prove it for three dimensions.

Lemma A.1. *Let $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ and assume u is homogeneous of degree 0, then*

$$\sum_{i,j=1}^3 u_{x_i x_j}^2(x) \geq 2|\nabla u(x)|^2 \quad \text{for any } x \in S^{n-1}.$$

Proof. If λ_i are the eigenvalues of the Hessian,

$$\sum_{i,j=1}^3 u_{x_i x_j}^2(x) = \sum_{i=1}^3 \lambda_i^2,$$

which is invariant under rotation. Therefore, we may assume without loss of generality that $x_0 \in S^{n-1}$ is $x_0 = (1, 0, 0)$, so that under spherical coordinates $\theta = 0$ and $\phi = \pi/2$. One may then explicitly compute that at x_0 we have

$$\begin{aligned} u_{xx}(x_0) &= 0 & u_{yy}(x_0) &= u_{\theta\theta}(x_0) & u_{xy}(x_0) &= -u_\theta(x_0) \\ u_{zz}(x_0) &= u_{\phi\phi}(x_0) & u_{xz}(x_0) &= u_\phi(x_0) & u_{yz}(x_0) &= -u_{\theta\phi}(x_0). \end{aligned}$$

Then at x_0 we obtain

$$\sum_{i,j=1}^3 u_{x_i x_j}^2 = u_{\theta\theta}^2 + u_{\phi\phi}^2 + 2u_{\theta\phi}^2 + 2u_\theta^2 + 2u_\phi^2.$$

Since at $x_0 = (1, 0, 0)$ we have

$$|\nabla u(x_0)|^2 = |\nabla_\theta f(x_0)|^2 = f_\theta^2 + f_\phi^2,$$

we conclude that

$$\sum_{i,j=1}^3 u_{x_i x_j}^2(x) \geq 2|\nabla u(x)|^2 = 2|\nabla_\theta f(x)|^2 \quad \text{for any } x \in S^{n-1}.$$

\square

We also have the following

Lemma A.2. *Let $f : S^2 \rightarrow \mathbb{R}$ with $u = r^\alpha f$ such that $\Delta u = 0$ in $\{u > 0\}$. If $0 < \alpha \leq 1$, then $|\nabla_\theta f|^2$ achieves its maximum on $S^{n-1} \cap \partial\{u > 0\}$.*

Proof. Since $\Delta u = 0$, then $\Delta_\theta f = -\alpha(\alpha + 1)f$. If $v = f$, then

$$\begin{aligned}\Delta|\nabla v|^2 &= 2\|D^2v\|^2 + 2\langle \nabla v, \nabla \Delta v \rangle \\ &= 2\|D^2v\|^2 - 2\alpha(\alpha + 1)|\nabla v|^2 \\ &\geq 0.\end{aligned}$$

The last inequality is a result of Lemma A.1. Then $|\nabla v|^2$ is subharmonic in $\{u > 0\}$, and consequently achieves the maximum on the boundary. Since $|\nabla v|^2 = |\nabla_\theta f|^2$, the conclusion of the Lemma is immediate. \square

APPENDIX B. C^1 CONVERGENCE ON A WEDGE-TYPE DOMAIN

In this appendix we show C^1 convergence of $w_\epsilon \rightarrow w_0$ on \overline{U} where w_ϵ, w_0, U are defined in the proof of Lemma 6.2. We recall that $U := \{x \in B_1 : x_3 > \cos \phi_2\}$ where $\phi_2 > \pi/2$ and was fixed in the proof of Lemma 6.2. It is clear that $w_\epsilon \rightarrow w_0$ in C^1 except at the corner $\{x \in \partial B_1 : x_3 = \cos \phi_2\}$. We handle the issue of the corner with a series of Lemmas.

Lemma B.1. *Let $V := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq \arctan(y/x) \leq \theta_0 < \pi\}$. If $\Delta u = 0$ and $u = 0$ on ∂V , and $|u| \leq C|x|$ for $|x| \geq 1$ and some $C > 0$, then $u \equiv 0$.*

Proof. Let $v(r, \theta, x_3) = r^\lambda \sin(\lambda\theta)$ where $\lambda\theta_0 = \pi$. Then $\Delta v = 0$ in V , $v = 0$ on ∂V , and $v \geq 0$ in V . Let w be the harmonic lifting of u_+ on $B_R \cap V$. We will choose R large and use the boundary Harnack principle [16].

$$\sup_{B_{R/2} \cap V} \frac{u_+}{v} \leq C_1 \sup_{B_{R/2} \cap v} \frac{w}{v} \leq \inf_{B_{R/2} \cap U} \frac{w}{v} \leq C_2 \frac{R}{R^\lambda}.$$

Since $\lambda > 1$, as $R \rightarrow \infty$ we obtain that $u_+ \equiv 0$. The same argument applies to u_- . \square

Lemma B.2. *Let $\Delta_c u = 0$ in U with $u = 0$ on $\{x_3 = \cos \phi_2\} \cap \overline{B_1}$ and $u \in C^1(\overline{U})$. Then there exists $C < \infty$ depending on $\|u\|_{C^1(\overline{U})}$ and c_0 but independent of c if $0 \leq c \leq c_0$ such that if $x_0 \in \partial B_1 \cap \{x_3 = \cos \phi_2\}$, then*

$$|u| \leq C|x - x_0|.$$

Proof. We first translate so that $x_0 = 0$ the origin. We will use compactness combined with a blow-up similar to the argument in the proof of Theorem 6.1 in [3]. Suppose by way of contradiction that no such C exists. Then there exists u_j, c_j, r_j with $\Delta_{c_j} u_j = 0$ and $r_j \rightarrow 0$ such that if

$$S_{r_j} = \sup_{\Omega \cap B_{r_j}} |u_j|$$

then

$$\begin{aligned}(i) \quad & \frac{S_{r_j}}{r_j} \rightarrow \infty \\ (ii) \quad & S_{r_j 2^k} \leq 2^k S_{r_j} \text{ for } k \in \mathbb{N} \text{ with } 2^k r_j \leq 1.\end{aligned}$$

We let

$$u_{r_j} := \frac{u(r_j x)}{S_{r_j}}.$$

We have the following

- (1) $\tilde{\Delta}_{c_j} u_{r_j} = 0$
- (2) $\sup_{B_{2^k}} |u_{r_j}| \leq 2^k$ whenever $2^k r_j \leq 1$
- (3) $u_{r_j}(0) = 0$
- (4) $\sup_{B_1} |u_{r_j}| = 1.$

Then $c_j \rightarrow c$, and $u_{r_j} \rightarrow u_0$ with

$$\begin{cases} a^{ij} \partial_{ij} u_0 = 0 \\ u_0 = 0 \text{ on } \partial U \\ u_0 \leq |x| \text{ for } |x| \geq 1, \end{cases}$$

where after rotation

$$a^{ij} = \frac{1}{(1+c^2)} \begin{pmatrix} 1+c^2 & 0 & 0 \\ 0 & 1+c^2 \cos^2 \phi_2 & -c^2 \sin \phi_2 \cos \phi_2 \\ 0 & -c^2 \sin \phi_2 \cos \phi_2 & 1+c^2 \sin^2 \phi_2 \end{pmatrix}.$$

Since a^{ij} is a constant coefficient matrix, a linear change of variables in only the x_2 and x_3 variables will give a new solution \tilde{u}_0 with $\Delta \tilde{u}_0 = 0$. For $0 \leq c \leq c_0$ with c_0 small, the transformed domain will still be a wedge-domain V with angle less than π . Then by Lemma B.1, it follows that $\tilde{u}_0 \equiv 0$ so that $u_0 \equiv 0$. This contradicts the fact that $\sup_{U \cap B_1} u_0 = 1$. □

Lemma B.3. *Let u_k be a sequence of solutions to $\Delta_{c_k} u_k = 0$ in U with $u_k = 0$ on $\{x_3 = \cos \phi_2\} \cap \overline{B_1}$. Assume $\lim_{k \rightarrow \infty} c_k = 0$ and r that $u_k \rightarrow v$ uniformly in \overline{U} and $u_k \rightarrow v$ in $C^1(\partial B_1 \cap \{x_3 \geq \phi_2\})$. Then $u_k \rightarrow v$ in $C^1(\overline{U})$.*

Proof. Suppose by way of contradiction that there exist points $x_k \in \partial U$ such that $x_k \rightarrow x_0$ and

$$(B.1) \quad \left| \frac{\partial u_k}{\partial \nu}(x_k) - \frac{\partial v}{\partial \nu}(x_k) \right| > \epsilon,$$

for some fixed ϵ . Let $|x_k - x_0| = r_k$. We rescale by

$$\tilde{u}_k = \frac{u_k(r_k(x - x_0))}{r_k}.$$

and

$$v_k := \frac{v(r_k(x - x_0))}{r_k}.$$

By Lemma B.2 we have that

$$|\tilde{u}_k|, |v_k| \leq C$$

Now $\tilde{u}_k - v_k \rightarrow v_0$ uniformly with $\Delta v_0 = 0$. Furthermore, since $u_k \rightarrow v$ in $C^1(\partial B_1 \cap \{x_3 \geq \phi_2\})$ it follows that $v_0 \equiv 0$ on ∂V where V is the domain obtained in the blowup and satisfies the assumptions of Lemma B.2. Then since v_0 has linear growth it follows from Lemma B.1 that $v \equiv 0$. Now from the C^1 convergence of $u_k - v_k$ away from the wedge of V and (B.1) it follows that there exists $x_0 \in \partial V \cap \partial B_1$ such that $|\nabla v_0| > \epsilon$. But this contradicts the fact that $v \equiv 0$. □

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